

On the Contact of Curves in a Four Dimensional Space.

by

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## On the Contact of Curves in a Four Dimensional Space.

### I. Introduction.

A theorem due to Halphen\* states "If two space curves (in a three dimensional space) have contact of order  $n$  at a point  $O$ , there exists a unique plane, called the principal tangent plane of the two curves, which passes through the common tangent and which has the property that the cones projecting the curves from any point  $P$  of this plane have contact of at least order  $n+1$  along the line  $PO$  ." Bompiani\*\* has recently shown (by methods distinct from those of Halphen) that the contact of the projecting cones will be of at least order  $n+2$  if  $P$  is on a unique line passing through  $O$  and lying in the principal plane and of least order  $n+3$  if  $P$  is a unique point on this line. The line and the point are called by Bompiani the principal line and the principal point.

More recently Stouffer\*\*\* has proven all three parts of the theorem by a method which is distinct from those of Halphen and Bompiani; his method will be used in this paper to study similar theorems in the four dimensional space.

The existence of the principal plane for two curves in

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\* Journal de L'Ecole Polytechnique, t. 47, 1880, pp. 25-27.

\*\* Memoire della R. Accademia della Scienze dell' Istituto di Bologna, Classe de Scienze Fisiche S. 8, t. 3, 1925-26, pp. 3-6.

\*\*\* An unpublished paper.

an  $n$ -space was demonstrated by Berzolari\*\* and later proven by R. G. Smith\*\*\*.

It is the purpose of this paper to prove, by Stouffer's method, the existence of the principal plane in a four dimensional space, and to study the properties of the plane and related problems.

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\*\*"Sugli invarianti differenziali proiettivi delle curve di un iperspazio" Ann. di Mat. s. II, t. XXVI, 1897.  
\*\*\*Doctors Dissertation, University of Kansas, 1930.

## II. Case I. Projections from a point.

Let two curves,  $\mathcal{C}$  and  $\mathcal{C}'$  lying in a four dimensional space but not in a three dimensional space, have order of contact  $n, n > 2$ . We shall assume that the equations of the two curves are expressed in projective non-homogeneous coordinates,  $x, y, z$ , and  $w$ , obtained from the projective homogeneous coordinates  $X_1, X_2, X_3, X_4, X_5$  by the relations:

$$x = \frac{X_2}{X_1}, \quad y = \frac{X_3}{X_1}, \quad z = \frac{X_4}{X_1}, \quad w = \frac{X_5}{X_1}.$$

Let the point of contact of the two curves be the origin; the common tangent, the  $x$ -axis; the common osculating plane, the  $x, y$  plane; and the common osculating three-space, the  $x, y, z$  three-space. Then the equations of the two curves,  $\mathcal{C}$  and  $\mathcal{C}'$ , in the neighborhood of the origin, may be put into the form:

For  $\mathcal{C}$ :

$$\begin{aligned} 1) \quad y &= l_2 x^2 + l_3 x^3 + \dots + l_n x^n + l_{n+1} x^{n+1} + l_{n+2} x^{n+2} + \dots \\ 2) \quad z &= m_3 x^3 + m_4 x^4 + \dots + m_n x^n + m_{n+1} x^{n+1} + m_{n+2} x^{n+2} + \dots \\ 3) \quad w &= p_4 x^4 + p_5 x^5 + \dots + p_n x^n + p_{n+1} x^{n+1} + p_{n+2} x^{n+2} + \dots \end{aligned}$$

For  $\mathcal{C}'$ :

$$\begin{aligned} 1) \quad y &= l'_2 x^2 + l'_3 x^3 + \dots + l'_n x^n + l'_{n+1} x^{n+1} + l'_{n+2} x^{n+2} + \dots \\ 2) \quad z &= m'_3 x^3 + m'_4 x^4 + \dots + m'_n x^n + m'_{n+1} x^{n+1} + m'_{n+2} x^{n+2} + \dots \\ 3) \quad w &= p'_4 x^4 + p'_5 x^5 + \dots + p'_n x^n + p'_{n+1} x^{n+1} + p'_{n+2} x^{n+2} + \dots \end{aligned}$$

where in general the prime coefficients of the equations of  $\mathcal{C}'$  are different from the corresponding coefficients of the equations of  $\mathcal{C}$ .

Since we assume that the order of contact of the two curves is not greater than  $n$ , at least one of the coefficients  $\rho_{n+1}, m_{n+1}, \rho_{n+1}$  is not equal to the corresponding prime. For the present, we shall assume  $\rho_{n+1} \neq \rho'_{n+1}$ . We shall now sketch briefly the general theory of our method of attack.

We determine the effect on the equations of  $\mathcal{C}$  and  $\mathcal{C}'$  of a transformation of coordinates which involves merely the transfer of the fifth vertex of the corresponding homogeneous coordinate system to a point whose coordinates are  $(a, b, c, d, 1)$ . The resulting equations in the new non-homogeneous variables  $X, Y, Z, W$  will be of the same form as equations (1), (2), (3) and (1'), (2'), (3'). The  $X, Y$ , and  $Z$  relations for the two curves will give us the projection of the two curves from the point  $(a, b, c, d, 1)$ , in the old homogeneous coordinate system. We shall attempt to determine  $a, b, c$  and  $d$  such that the order of contact of the projecting cones will be greater than  $n$ .

To determine the transformation of coordinates that will merely change the fifth vertex of the corresponding homogeneous coordinate system to a point whose coordinates are  $(a, b, c, d, 1)$ , we evaluate the coefficients in the general transformation:

$$X_i = a_i X_1 + b_i X_2 + c_i X_3 + d_i X_4 + e_i X_5, \quad i=1, 2, 3, 4, 5,$$

We find our transformation, in homogeneous coordinates, to be:

$$X_1 = X_1 - a X_5,$$

$$X_2 = X_2 - b X_5,$$

$$X_3 = X_3 - c X_5,$$

$$X_4 = x_4 - dx_3,$$

$$X_5 = x_5,$$

This transformation is expressed in non-homogeneous coordinates by:

$$4) X = \frac{x - bw}{1 - aw} = (x - bw)(1 + aw + a^2 w^2 + \dots),$$

$$5) Y = \frac{y - cw}{1 - aw} = (y - cw)(1 + aw + a^2 w^2 + \dots),$$

$$6) Z = \frac{z - dw}{1 - aw} = (z - dw)(1 + aw + a^2 w^2 + \dots),$$

$$7) W = \frac{w}{1 - aw} = (w)(1 + aw + a^2 w^2 + \dots),$$

In (4), we substitute for  $w$  the expression given by (3); the resulting equation expresses the relation between the coordinate of  $\mathcal{C}$  and  $x$ . We obtain:

$$8) X = [x - b(p_4 x^4 + \dots + p_n x^n + p_{n+1} x^{n+1} + \dots)] [1 + a(p_4 x^4 + \dots + p_n x^n + \dots) + a^2(p_4 x^4 + \dots + p_n x^n + p_{n+1} x^{n+1} + \dots) + \dots],$$

which upon simplifying becomes:

$$8) X = x - b p_4 x^4 + x^5 (-b p_5 + a p_4) + x^6 N_6 + \dots + x^{n+1} (-b p_{n+1} + N_{n+1}) + x^{n+2} (-b p_{n+2} + a p_{n+1} + N_{n+2}) + x^{n+3} (-b p_{n+3} + a p_{n+2} + N_{n+3}) + \dots,$$

where  $N_f$  is that part of the coefficient of  $x^f$  that does not contain  $m$  or  $\rho$  with a subscript greater than  $n$ .

Similarly we find the equation expressing the relation between the  $Y$  coordinate of  $\mathcal{C}$  and  $x$ . It is:

$$9) Y = l_2 x^2 + l_3 x^3 + x^4 (l_4 - c p_4) + x^5 M_5 + \dots + x^n M_n + x^{n+1} (l_{n+1} - c p_{n+1} + M_{n+1}) + x^{n+2} (l_{n+2} - c p_{n+2} + M_{n+2}) + x^{n+3} (l_{n+3} - c p_{n+3} + a l_2 p_{n+1} + M_{n+3}) + \dots,$$

where  $M_g$  is that part of the coefficient of  $x^g$  that does not contain  $l$  or  $p$  with a subscript greater than  $n$ .

Also we find the equation expressing the relation between the  $Z$  coordinate of  $\mathcal{C}$  and  $x$ . It is:

$$(10) \quad Z = m_3 x^3 + x^4 (m_4 - d/p) + x^5 p_5 + \dots + x^n p_n + x^{n+1} (m_{n+1} - d/p_{n+1} + p_{n+1}) \\ + x^{n+2} (m_{n+2} - d/p_{n+2} + p_{n+2}) + x^{n+3} (m_{n+3} - d/p_{n+3} + p_{n+3}) + \dots,$$

where  $p_g$  is that part of the coefficient of  $x^g$  that does not contain  $m$  or  $p$  with a subscript greater than  $n$ .

Considering  $x$  as a parameter, equations (8), (9), and (10) are the equations of a cone, the projection of the curve  $\mathcal{C}$  from the point  $(a b c d 1)$  in the original corresponding homogeneous coordinate system. To eliminate  $x$  between equations (8) and (9) we first square (8), multiply by  $l_2$  and subtract from (9). We then have:

$$(9^2) \quad Y - l_2 X^2 = l_3 x^3 + M'_4 x^4 + \dots + x^n M'_n + x^{n+1} (l_{n+1} - c p_{n+1} + M'_{n+1}) \\ + x^{n+2} (l_{n+2} - c p_{n+2} + 2 l_2 b p_{n+1} + M'_{n+2}) \\ + x^{n+3} (l_{n+3} - c p_{n+3} + a l_2 p_{n+1} + 2 l_2 b p_{n+2} - 2 l_2 a p_{n+1} + M'_{n+3}) + \dots,$$

where  $M'_g$  has the same significance as  $M_g$ .

We now cube (8), multiply by  $l_3$  and subtract from (9<sup>2</sup>).

We obtain:

$$(9^3) \quad Y - l_2 X^2 - l_3 X^3 = x^4 M''_4 + x^5 M''_5 + \dots + x^n M''_n + x^{n+1} (l_{n+1} - c p_{n+1} + M''_{n+1}) \\ + x^{n+2} (l_{n+2} - c p_{n+2} + 2 l_2 b p_{n+1} + M''_{n+2}) \\ + x^{n+3} (l_{n+3} - c p_{n+3} + a l_2 p_{n+1} + 2 l_2 b p_{n+2} - 2 l_2 a p_{n+1} + 3 l_3 b p_{n+1} + M''_{n+3}) + \dots,$$

where  $M_f''$  has the same significance as  $M_f$ .

We continue this process indefinitely. We notice, however, that if we raise (for example) (8) to the fourth power, multiply by a suitable factor and subtract from (9<sup>3</sup>), the coefficients of  $x^{n+1}$  in (8) first appears in the coefficient of  $x^{n+4}$  of the resulting equation. Such considerations lead us to the concept:

We denote by (9<sup>m</sup>) the equation in  $Y$ ,  $X$ , and  $x$  which we obtain our process of elimination by that step in which we raise equation (8) to the  $m$ -th power, multiply by a suitable factor and subtract from the preceding  $X$ ,  $Y$ ,  $x$  equation [which we may now denote by (9)<sup>m-1</sup>]. Then the coefficient of  $X^n$  in (9<sup>m</sup>) is obviously the coefficient of  $x^m$  in (9<sup>m-1</sup>).

Since the raising of equation (8) to any power greater than three does not bring to the coefficients of  $x^{n+1}$ ,  $x^{n+2}$ ,  $x^{n+3}$  any terms involving  $l$ ,  $m$ , or  $\rho$  with a subscript greater than  $n$  we may say that the coefficients of  $X^{n+1}$ ,  $X^{n+2}$ ,  $X^{n+3}$  in (8<sup>q</sup>) ( $q > 3$ ) are the same as in (8<sup>3</sup>) save for additive constants which do not involve  $l$ ,  $m$ , or  $\rho$  with subscript greater than  $n$ . Then in our final equation, obtained by eliminating  $x$  from (8) and (9), the coefficients of  $X^{n+1}$ ,  $X^{n+2}$ ,  $X^{n+3}$  will be the same as the corresponding coefficients in (8<sup>3</sup>), plus terms not involving  $l$ ,  $m$ , or  $\rho$  with subscript greater than  $n$ .

We may then express the relation between  $Y$  and  $X$  as:

$$11) Y = \lambda_2 X^2 + \lambda_3 X^3 + \dots + \lambda_n X^n + \lambda_{n+1} X^{n+1} + \lambda_{n+2} X^{n+2} + \lambda_{n+3} X^{n+3} + \dots,$$



where:

$$11-a) \lambda_{n+1} = l_{n+1} - c p_{n+1} + M_{n+1}'' ,$$

$$11-b) \lambda_{n+2} = l_{n+2} - c p_{n+2} + 2 l_2 l_{n+1} + M_{n+2}'' ,$$

$$11-c) \lambda_{n+3} = l_{n+3} - c p_{n+3} - a l_2 p_{n+1} - 2 l_2 l_{n+2} + 3 l_3 l_{n+1} + M_{n+3}'' ,$$

where  $M_g''$  has the same significance as  $M_g$  .

Using the same principle, we eliminate  $x$  from (8) and (10). Our final  $Z$ ,  $X$  relation is:

$$12) Z = \mu_3 X^3 + \mu_4 X^4 + \dots + \mu_n X^n + \mu_{n+1} X^{n+1} + \mu_{n+2} X^{n+2} + \mu_{n+3} X^{n+3} + \dots ,$$

where:

$$12-a) \mu_{n+1}' = m_{n+1} - d p_{n+1} + p_{n+1}'' ,$$

$$12-b) \mu_{n+2}' = m_{n+2} - d p_{n+2} + p_{n+2}'' ,$$

$$12-c) \mu_{n+3}' = m_{n+3} - d p_{n+3} + 3 m_3 l_{n+1} + p_{n+3}'' ,$$

where  $p_g''$  has the same significance as  $p_g$  .

Equations (11) and (12) are the equations of a cone (in four-dimensional space); the projection of the curve  $\mathcal{C}$  from the point  $(a, b, c, d)$  in the old homogeneous coordinate system. We shall next obtain the equations (in the new, non-homogeneous coordinate system) of the cone; the projection of  $\mathcal{C}'$  from the same point. We greatly facilitate our work if we observe that corresponding equations for  $\mathcal{C}'$  are obtained in exactly the same manner as for  $\mathcal{C}$ , therefore the only difference in the two sets of equations will be that  $l$ ,  $m$ , and  $p$  (in the  $\mathcal{C}'$  set), with subscripts greater than  $n$  will appear as  $l'$ ,  $m'$  and  $p'$  with the same subscripts, respect-

ively. Since we have already noted those terms involving  $l, m$ , or  $p$  with subscripts greater than  $n$ , we may immediately write down the equations of the cone, which is the projection of  $\mathcal{C}'$  from the point  $(a, b, c, d)$ .

They are:

$$(13) Y = \lambda_2 X^2 + \lambda_3 X^3 + \dots + \lambda_n X^n + \lambda'_{n+1} X^{n+1} + \lambda'_{n+2} X^{n+2} + \lambda'_{n+3} X^{n+3} + \dots$$

$$(14) Z = \mu_3 X^3 + \mu_4 X^4 + \dots + \mu_n X^n + \mu'_{n+1} X^{n+1} + \mu'_{n+2} X^{n+2} + \mu'_{n+3} X^{n+3} + \dots$$

where:

$$(13-a) \lambda'_{n+1} = l'_{n+1} - c p'_{n+1} + M'''_{n+1},$$

$$(13-b) \lambda'_{n+2} = l'_{n+2} - c p'_{n+2} + 2 l_2 p'_{n+1} + M'''_{n+2},$$

$$(13-c) \lambda'_{n+3} = l'_{n+3} - c p'_{n+3} + 6 l_3 p'_{n+1} + 2 l_2 p'_{n+2} - 2 l_2 a p'_{n+1} + 3 l_3 p'_{n+1} + M'''_{n+3},$$

and:

$$(14-a) \mu'_{n+1} = m'_{n+1} - c p'_{n+1} + P''_{n+1},$$

$$(14-b) \mu'_{n+2} = m'_{n+2} - c p'_{n+2} + P''_{n+2},$$

$$(14-c) \mu'_{n+3} = m'_{n+3} - c p'_{n+3} + 3 m_3 p'_{n+1} + P''_{n+3}.$$

We proceed to study the order of contact of the two curves, the one defined by equations (11) and (12); the other defined by equations (13) and (14). From (13) and (14) we notice that

$$\lambda'_{n+1} = \lambda_{n+1} \quad \text{if and only if:}$$

$$l'_{n+1} - c p'_{n+1} + M'''_{n+1} = l_{n+1} - c p_{n+1} + M'''_{n+1} \quad \text{or}$$

$$(16) c = \frac{l_{n+1} - l'_{n+1}}{p_{n+1} - p'_{n+1}}.$$

We also notice that  $\frac{m'_{n+1}}{\rho'_{n+1}} = \frac{m_{n+1}}{\rho_{n+1}}$  if and only if:

$$m'_{n+1} - c/\rho'_{n+1} + \rho''_{n+1} = m_{n+1} - c/\rho_{n+1} + \rho''_{n+1}, \text{ or}$$

$$d' = \frac{m_{n+1} - m'_{n+1}}{\rho_{n+1} - \rho'_{n+1}}.$$

Since we have assumed that  $\rho_{n+1} \neq \rho'_{n+1}$  we may conclude that if we assign certain values to  $c$  and  $d$ , the cones projecting  $C$  and  $C'$  from  $(abcd)$  have contact of order  $n+1$ . Since we have placed no restriction on  $a$  and  $b$ , we have two degrees of freedom in which to choose the vertex of our cones and have order of contact  $n+1$ . Then the vertex of the cones may vary over a plane determined by the variations of  $a$  and  $b$ . Our theorem then follows:

Theorem: "If two four dimensional curves have contact of order  $n$  at a point  $O$  there exists a unique plane, called the principal tangent plane, which has the property that the cones projecting the curves from any point  $P$  of this plane have contact of order at least  $n+1$  along the line  $PO$ ."

We next show that the principal tangent plane contains the common tangent, at  $O$ , of the two curves.

The equation of a three space containing the first and second vertices of our coordinate system and any two other points with coordinates  $(l, l_1, l_2, l_3)$  and  $(l', l'_1, l'_2, l'_3)$  is:

$$\begin{vmatrix}
 X_1 & X_2 & X_3 & X_4 & X_5 \\
 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 \\
 l_1 & l_2 & l_3 & l_4 & l_5 \\
 l_1' & l_2' & l_3' & l_4' & l_5'
 \end{vmatrix} = 0$$

On expanding, this equation reduces to:

$$X_3 (l_4 l_5' - l_4' l_5) + X_4 (l_3 l_5' - l_3' l_5) + X_5 (l_3 l_4' - l_3' l_4) = 0, \text{ or}$$

$$K_1 X_3 + K_2 X_4 + K_3 X_5 = 0.$$

Therefore, the equation of any three-space containing the first and second vertices has the form:

$$L_1 X_3 + L_2 X_4 + L_3 X_5 = 0,$$

where  $L_1$ ,  $L_2$  and  $L_3$  are general constants.

The equations of a plane containing the first and second vertices are merely the equations of any two non-coincident three spaces containing the first and second vertices. We may therefore write down the equations of a plane containing the two vertices as:

$$\begin{aligned}
 a) \quad & H X_3 + B X_4 + C X_5 = 0, \\
 & \alpha X_3 + \beta X_4 + \gamma X_5 = 0,
 \end{aligned}$$

where  $H$ ,  $B$ ,  $C$  are not proportional to  $\alpha$ ,  $\beta$ ,  $\gamma$ .

To come back to a discussion of the location of the fifth vertex in our problem; we have assigned values to  $c$  and  $d$ , allowing  $a$  and  $b$  to vary. If we substitute these values of  $c$ ,  $d$ , and  $1$  for  $x_1$ ,  $x_4$  and  $x_5$  respectively in equations (a) we obtain:

$$a) \quad \alpha c + \beta d + \gamma = 0,$$

$$c) \quad \alpha c + \beta d + \gamma = 0,$$

We may find values of  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\alpha$ ,  $\beta$ ,  $\gamma$  that will satisfy (a) and (c); therefore we may conclude that we are able to find two three-spaces that contain our first two vertices and the point  $(a b c d 1)$ , regardless of the values of  $a$  and  $b$ . Then, the point  $(a b c d 1)$ , where  $a$  and  $b$  may vary, is in a plane which contains the first two vertices of our homogeneous coordinate system; thus it contains the common tangent to our two curves. The second part of our theorem then follows:

Theorem: "The principal tangent plane of two curves,  $C$  and  $C'$ , in a four dimensional space contains the common tangent of the two curves."

From equations (11-b) and (13-b) we see that we may assign a value to  $b$  such that  $\lambda_{n+2} = \lambda'_{n+2}$ . In general, however, the inequality  $\lambda_{n+2} \neq \lambda'_{n+2}$  is true, regardless of the choice of the fifth vertex. Then in general, and without regard to the point of projec-

tion, the projection of two curves of order of contact  $n$  cannot have contact of order greater than  $n+1$ . Hence, in the principal plane there is no "principal line", and no "principal point" in a sense analogous to that of the three dimensional case.

Equations (12-a, 12-b) and (14-a, 14-b) tell us that

$$\mu_{n+1} = \mu'_{n+1}, \text{ and } \mu_{n+2} = \mu'_{n+2},$$

if and only if:

$$m'_{n+1} - d p'_{n+1} = m_{n+1} - d p_{n+1}, \quad \text{and}$$

$$m'_{n+2} - d p'_{n+2} = m_{n+2} - d p_{n+2},$$

or:

$$d = \frac{m_{n+1} - m'_{n+1}}{p_{n+1} - p'_{n+1}}, \quad \text{and}$$

$$d = \frac{m_{n+2} - m'_{n+2}}{p_{n+2} - p'_{n+2}}.$$

These conditions are equivalent to:

$$\Delta = \begin{vmatrix} m_{n+1} - m'_{n+1} & p_{n+1} - p'_{n+1} \\ m_{n+2} - m'_{n+2} & p_{n+2} - p'_{n+2} \end{vmatrix} = 0$$

If  $\Delta = 0$ , we may obtain projections of our two curves  $\mathcal{C}$  and  $\mathcal{C}'$  which have order of contact  $n+2$ , by assigning values to  $a$ ,  $b$ ,  $c$ , and  $d$ . Thus in this very special case there

exists a unique line on the principal tangent plane; this line possesses the property that the projections of the two curves from any point on the line have contact of order  $n+2$ . We may define this line as the "principal line" of our  $\mathcal{C}$  and  $\mathcal{C}'$  curves. We cannot overemphasize the fact that in this case  $\mathcal{C}$  and  $\mathcal{C}'$  are special curves. We show in the next paragraphs that the principal line contains the point of contact of the two curves.

Our problem is this: we have a point whose homogeneous coordinates are  $(a b c d 1)$ ;  $b, c, d$  are fixed;  $a$  varies, thus the point describes a straight line. We are to show that this straight line contains the first vertex  $(10000)$  of our coordinate system.

The equation of any three space containing the point  $(10000)$  is:

$$H_2 x_2 + H_3 x_3 + H_4 x_4 + H_5 x_5 = 0,$$

The equations of our principal plane are:

$$A x_3 + B x_4 + C x_5 = 0,$$

$$\alpha x_3 + \beta x_4 + \gamma x_5 = 0,$$

Then the equations of a line in the principal plane, through the point  $(10000)$  are:

$$d) H_2 x_2 + H_3 x_3 + H_4 x_4 + H_5 x_5 = 0,$$

$$e) A x_3 + B x_4 + C x_5 = 0,$$

$$f) \alpha x_3 + \beta x_4 + \gamma x_5 = 0,$$

Where  $A, B, C, \alpha, \beta, \gamma$  are limited constants, and the  $K_s$  are variable constants, depending upon the choice of our line. We substitute in the equations for  $x_2, x_3, x_4, x_5$  the values  $l, c, d, 1$  respectively:

$$g) K_2 b + K_3 c + K_4 d + K_5 = 0,$$

$$h) A c + B d + C = 0,$$

$$i) \alpha c + \beta d + \gamma = 0,$$

Equations  $h$  and  $i$  are immediately satisfied as, from previous work, they are identities. Considering equation  $(g)$  we may assign to  $K_4$  and  $K_5$  independent values and solve for the values of  $K_2$ , and  $K_3$  that will satisfy  $(g)$ . Thus our coordinates may satisfy equations  $g, h$ , and  $i$ , regardless of the value of  $a$ , and therefore our principal line contains the point  $(10000)$ , which is the point of contact of the two curves.

We may question whether it might not be possible to transform our other vertices of our polygon of reference so as to put the equations of the curves in a form such that  $\Delta = 0$ . Geometrically, we may see the impossibility of this when we realize that any transformation of our first four vertices is merely a transformation of coordinates in the common osculating three space of the two curves, hence it cannot affect the order of contact of the projections from a point outside the three-space. Analytically, we find that  $\Delta$  is an absolute invariant under the transforms



of all five vertices. The analytic work is not reproduced here.

The restrictions on the two curves necessary to have

$\lambda_{H+3} = \lambda'_{H+3}$  and  $\mu_{H+3} = \mu'_{H+3}$  are too complex to be of much interest, and are not discussed here.

### III. Special Cases.

Our work thus far is based upon the assumption that in equations (3) and (3')  $\rho_{n+1} \neq \rho'_{n+1}$ . It is possible to have two curves whose equations are such that  $\rho_{n+1} = \rho'_{n+1}$ . We now show that the theorems already proved hold also for this case.

Assume  $\rho_{n+1} = \rho'_{n+1}$ . Then equations (11), (12), (13) and (14), and the work preceding them, tell us that it is impossible to have projections of  $\mathcal{C}$  and  $\mathcal{C}'$  of order of contact greater than  $n$  from any point not in the common osculating three space. We then investigate the projections from a point in this osculating three-space. To do this we make a transformation of coordinates that involves merely the transformation of the fourth vertex of the corresponding homogeneous coordinate system to the point  $(e f g 1 0)$

Evaluating the coefficients of our general transformation as before, we obtain our transformation in the form:

$$X_1 = x_1 - e x_4,$$

$$X_2 = x_2 - f x_4,$$

$$X_3 = x_3 - g x_4,$$

$$X_4 = x_4,$$

$$X_5 = x_5,$$

The equations may be expressed in the corresponding non-homogeneous coordinate system as:

$$18) X = \frac{x - fz}{1 - \epsilon z} = (x - fz)(1 + \epsilon z + \epsilon^2 z^2 + \dots),$$

$$19) Y = \frac{y - gz}{1 - \epsilon z} = (y - gz)(1 + \epsilon z + \epsilon^2 z^2 + \dots),$$

$$20) Z = \frac{z}{1 - \epsilon z} = (z)(1 + \epsilon z + \epsilon^2 z^2 + \dots),$$

$$21) W = \frac{w}{1 - \epsilon z} = (w)(1 + \epsilon z + \epsilon^2 z^2 + \dots),$$

We are to study the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from the point  $(efg10)$  in our old coordinate system; which is the fourth vertex of our new coordinate system. Applying the same theory as that which we used to study the projections from the fifth vertex, we note that the equations of the projections from  $(efg10)$  of  $\mathcal{C}$  and  $\mathcal{C}'$  are obtained as the transformed  $X$ ,  $Y$ , and  $W$  relations of the equations  $\mathcal{C}$  and  $\mathcal{C}'$  respectively. Analytically, the work involved is similar to that used in finding the projections from the fifth vertex. Our final results are:

Our  $X$ ,  $Y$ , and  $W$  transformation equations of  $\mathcal{C}$  are:

$$22) Y = \lambda_2 X^2 + \lambda_3 X^3 + \dots + \lambda_n X^n + \lambda_{n+1} X^{n+1} + \lambda_{n+2} X^{n+2} + \lambda_{n+3} X^{n+3} + \dots,$$

$$23) W = \nu_4 X^4 + \nu_5 X^5 + \dots + \nu_n X^n + \nu_{n+1} X^{n+1} + \nu_{n+2} X^{n+2} + \nu_{n+3} X^{n+3} + \dots,$$

where:

$$22-a) \lambda_{n+1} = l_{n+1} - g m_{n+1} + M_{n+1}^{II},$$

$$22-b) \lambda_{n+2} = l_{n+2} - g m_{n+2} + 2l_2 f m_{n+1} + M_{n+2}^{II}$$

$$22-c) \lambda_{n+3} = l_{n+3} - g m_{n+3} - l_2 \epsilon m_{n+1} + 2l_2 f m_{n+1} + 3(l_3 - g m_3) f m_{n+1} + M_{n+3}^{II},$$

where  $M_f^{\text{II}}$  is analogous to  $M_f$ .

and where:

$$23-a) \gamma_{n+1} = \rho_{n+1} + R_{n+1},$$

$$23-b) \gamma_{n+2} = \rho_{n+2} + R_{n+2},$$

$$23-c) \gamma_{n+3} = \rho_{n+3} + R_{n+3},$$

where  $R_f$  is analogous to  $M_f$  and  $\rho_f$ .

Applying  $X$ ,  $Y$ , and  $W$  transformations to  $\mathcal{C}'$  we obtain:

$$24) Y = \lambda_1 X^1 + \lambda_2 X^2 + \dots + \lambda_n X^n + \lambda'_{n+1} X^{n+1} + \lambda'_{n+2} X^{n+2} + \lambda'_{n+3} X^{n+3} + \dots,$$

$$25) W = \nu_1 X^1 + \nu_2 X^2 + \dots + \nu_n X^n + \nu'_{n+1} X^{n+1} + \nu'_{n+2} X^{n+2} + \nu'_{n+3} X^{n+3} + \dots,$$

where:

$$24-a) \lambda'_{n+1} = l'_{n+1} - g m'_{n+1} + M_{n+1}^{\text{II}},$$

$$24-b) \lambda'_{n+2} = l'_{n+2} - g m'_{n+2} + 2l_2 f m'_{n+1} + M_{n+2}^{\text{II}},$$

$$24-c) \lambda'_{n+3} = l'_{n+3} - g m'_{n+3} - l_2 m'_{n+1} + 2l_2 f m'_{n+1} + 3(l_2 - g m_3) f m'_{n+1} + M_{n+3}^{\text{II}},$$

and

$$25-a) \nu'_{n+1} = \rho'_{n+1} + R_{n+1},$$

$$25-b) \nu'_{n+2} = \rho'_{n+2} + R_{n+2},$$

$$25-c) \nu'_{n+3} = \rho'_{n+3} + R_{n+3},$$

Remembering that equations (21) and (23) are the equations of the projection from  $(\epsilon f g \mid u)$  of  $\mathcal{C}$ , and that (24) and (25) are the equations of the projection from  $(\epsilon f g \mid u)$  of  $\mathcal{C}'$  we

proceed to study the order of contact of the two projections.

Assume  $m_{n+1} \neq m'_{n+1}$ . We see from equations (22-a) and (24-a)

that we may assign a value to  $g$  such that  $\lambda_{n+1} = \lambda'_{n+1}$ . Since

$\rho_{n+1} = \rho'_{n+1}$  by assumption, and therefore  $\gamma_{n+1} = \gamma'_{n+1}$ , we have that

for such a value of  $g$  the projections of the two curves from

the point  $(\epsilon, g, 1, 0)$  is of order of contact  $n+1$ . Since  $\epsilon$  and

$g$  may vary we have again determined a "principal tangent plane."

By the identical reasoning we used before we may show that this

principal tangent plane contains the tangent line. In order for

our projections to have contact of order  $n+2$ ,  $\rho_{n+2} = \rho'_{n+2}$ .

As before the principal line so determined (by the variations of

$\epsilon$ ) contains the point of contact of the two curves.

There still remains the exceptional case where  $\rho_{n+1} = \rho'_{n+1}$

and  $m_{n+1} = m'_{n+1}$ . Under these conditions our original equations

(2, 3) and (2', 3) of  $\mathcal{C}$  and  $\mathcal{C}'$  tell us immediately that the pro-

jections, from the third vertex of the corresponding homogeneous

coordinate system, of our two curves are of order of contact  $n+1$ .

Since the only limitation on the third vertex is that it be in the

common osculating plane of the two curves, our common osculating

plane is our principal tangent plane. It contains the tangent line.

To determine the conditions for the existence of a principal line

we must make a transformation of coordinates that involves merely

the transform of the third vertex of the corresponding homogeneous

coordinate system to a point whose coordinates are  $(h, i, 1, 0, 0)$ . If

we submit  $\mathcal{C}$  and  $\mathcal{C}'$  to the transformation the resulting  $X$ ,  $Z$ , and  $W$  relations are the equations of the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from  $(h i l o o)$ . The analytic work is similar to that used in finding the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from the point  $(a b c d l)$  by a transformation of the fifth vertex. Our final results are:

Our  $X$ ,  $Z$ , and  $W$  transformed equations of  $\mathcal{C}$  are:

$$26) Z = \mu_3 X^3 + \mu_4 X^4 + \dots + \mu_n X^n + \mu_{n+1} X^{n+1} + \mu_{n+2} X^{n+2} + \mu_{n+3} X^{n+3} + \dots,$$

$$27) W = \nu_4 X^4 + \nu_5 X^5 + \dots + \nu_n X^n + \nu_{n+1} X^{n+1} + \nu_{n+2} X^{n+2} + \nu_{n+3} X^{n+3} + \dots,$$

where

$$26-a) \mu_{n+1} = M_{n+1} + R_{n+1},$$

$$26-b) \mu_{n+2} = M_{n+2} + R_{n+2},$$

$$26-c) \mu_{n+3} = M_{n+3} + 2i \int_{n+1} + R_{n+3},$$

where  $R_j$  is analogous to  $M_j$ ,  $P_j$ , etc.

and where

$$27-a) \nu_{n+1} = \rho_{n+1} + S_{n+1},$$

$$27-b) \nu_{n+2} = \rho_{n+2} + S_{n+2},$$

$$27-c) \nu_{n+3} = \rho_{n+3} + S_{n+3},$$

Our  $X$ ,  $Z$ , and  $W$  transformed equations of  $\mathcal{C}'$  are:

$$28) Z = \mu'_3 X^3 + \mu'_4 X^4 + \dots + \mu'_n X^n + \mu'_{n+1} X^{n+1} + \mu'_{n+2} X^{n+2} + \mu'_{n+3} X^{n+3} + \dots,$$

$$29) W = \nu'_4 X^4 + \nu'_5 X^5 + \dots + \nu'_n X^n + \nu'_{n+1} X^{n+1} + \nu'_{n+2} X^{n+2} + \nu'_{n+3} X^{n+3} + \dots,$$

where:

$$28-a) \mu'_{n+1} = m'_{n+1} + R_{n+1},$$

$$28-b) \mu'_{n+2} = m'_{n+2} + R_{n+2},$$

$$28-c) \mu'_{n+3} = m'_{n+3} + 2i \rho'_{n+1} + R_{n+3},$$

and where

$$29-a) \nu'_{n+1} = \rho'_{n+1} + S_{n+1},$$

$$29-b) \nu'_{n+2} = \rho'_{n+2} + S_{n+2},$$

$$29-c) \nu'_{n+3} = \rho'_{n+3} + S_{n+3},$$

These equations tell us that the necessary and sufficient conditions for the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from the point ( $h:1:0:0$ ) to have contact of order  $n+2$  are that  $m_{n+2} = m'_{n+2}$  and  $\rho_{n+2} = \rho'_{n+2}$ .

When these conditions hold the projections from any point in the principal tangent plane will have contact of order  $n+2$ ; thus even under these conditions there is no "principal line."

We have yet to consider the exceptional cases where  $n$  is less than 2. In case  $n=1$ ,  $\mathcal{C}$  and  $\mathcal{C}'$  no longer have a common osculating three space at the point of contact; then their equations have the form:

For  $\mathcal{C}$ :

$$y = l_1 x^2 + l_2 x^3 + \dots,$$

$$z = m_1 x^2 + m_2 x^3 + \dots,$$

$$w = \rho_1 x^3 + \rho_2 x^4 + \dots$$

For  $\mathcal{C}'$  :

$$y = l_2' x^2 + l_3' x^3 + \dots,$$

$$z = m_3' x^3 + m_4' x^4 + \dots,$$

$$w = p_3' x^3 + p_4' x^4 + \dots$$

The equations of the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from the point  $(abcd)$  are:

For  $\mathcal{C}$  :

$$Y = \lambda_1 X^2 + \lambda_3 X^3 + \lambda_4 X^4 + \lambda_5 X^5 + \dots,$$

$$Z = \mu_3 X^3 + \mu_4 X^4 + \mu_5 X^5 + \dots$$

For  $\mathcal{C}'$  :

$$Y = \lambda_1' X^2 + \lambda_3' X^3 + \lambda_4' X^4 + \lambda_5' X^5 + \dots,$$

$$Z = \mu_3' X^3 + \mu_4' X^4 + \mu_5' X^5 + \dots$$

where, if we denote by  $\Delta R$  the difference between  $R$  and  $R'$  :

$$\Delta \lambda_3 = \Delta l_3 - c \Delta p_3,$$

$$\Delta \lambda_4 = \Delta l_4 - c \Delta p_4 + 2l_2 b \Delta p_3,$$

$$\Delta \lambda_5 = \Delta l_5 - c \Delta p_5 + a \Delta p_3 l_2 + 2l_2 b \Delta p_4 - 2l_2 a \Delta p_3 + 3(\Delta l_3 - c \Delta p_3) b \Delta p_3,$$

$$\begin{aligned} \Delta \lambda_6 = & \Delta l_6 - c \Delta p_6 + a \Delta p_3 (\Delta l_3 - c \Delta p_3) + a \Delta p_4 l_2 + 2l_2 b \Delta p_5 - 2l_2 a \Delta p_4 \\ & - l_2 b^2 \Delta p_3^2 + 3(\Delta l_3 - c \Delta p_3) (b \Delta p_4 - a \Delta p_3) + 6(\Delta l_4 - c \Delta p_4 + 2l_2 b \Delta p_3) \Delta p_3, \end{aligned}$$



and:

$$\Delta \mu_3 = \Delta m_3 - d\Delta p_3,$$

$$\Delta \mu_4 = \Delta m_4 - d\Delta p_4,$$

$$\Delta \mu_5 = \Delta m_5 - d\Delta p_5 + 3(\Delta m_3 - d\Delta p_3)\Delta p_5,$$

$$\begin{aligned} \Delta \mu_6 = \Delta m_6 - d\Delta p_6 + d\Delta p_5(\Delta m_3 - d\Delta p_3) - 3(\Delta m_3 - d\Delta p_3)(-\Delta p_4 + d\Delta p_5) \\ + K(\Delta m_4 - d\Delta p_4)\Delta p_5, \end{aligned}$$

We see from these equations that the theorems already proved for the general case hold for the case where  $k = 2$ .

In the exceptional case where  $k = 1$ ,  $\mathcal{C}$  and  $\mathcal{C}'$  no longer have a common osculating plane at the point of contact; their equations may then have the form:

For  $\mathcal{C}$ :

$$y = l_2 x^2 + l_3 x^3 + \dots,$$

$$z = m_2 x^2 + m_3 x^3 + \dots,$$

$$w = p_2 x^2 + p_3 x^3 + \dots,$$

For  $\mathcal{C}'$ :

$$y = l'_2 x^2 + l'_3 x^3 + \dots,$$

$$z = m'_2 x^2 + m'_3 x^3 + \dots,$$

$$w = p'_2 x^2 + p'_3 x^3 + \dots,$$

The equations of the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from the point  $(a b c d 1)$  are:

For  $\mathcal{C}$  :

$$Y = \lambda_2 X^2 + \lambda_3 X^3 + \lambda_4 X^4 + \dots,$$

$$Z = \mu_2 X^2 + \mu_3 X^3 + \mu_4 X^4 + \dots,$$

For  $\mathcal{C}'$  :

$$Y = \lambda_2' X^2 + \lambda_3' X^3 + \lambda_4' X^4 + \dots,$$

$$Z = \mu_2' X^2 + \mu_3' X^3 + \mu_4' X^4 + \dots,$$

where:

$$\Delta \lambda_2 = \Delta \rho_2 - c \Delta \rho_2,$$

$$\Delta \lambda_3 = \Delta \rho_3 - c \Delta \rho_3 + 2(\Delta \rho_2 - c \Delta \rho_2) b \Delta \rho_2,$$

$$\begin{aligned} \Delta \lambda_4 = & \Delta \rho_4 - c \Delta \rho_4 + a \Delta \rho_2 (\Delta \rho_2 - c \Delta \rho_2) + 2(\Delta \rho_2 - c \Delta \rho_2) (b \Delta \rho_3 - a \Delta \rho_2) \\ & + (\Delta \rho_2 - c \Delta \rho_2) b^2 \overline{\Delta \rho_2}^2 + 3b \Delta \rho_2 [\Delta \rho_3 - c \Delta \rho_3 + 2(\Delta \rho_2 - c \Delta \rho_2) b \Delta \rho_2], \end{aligned}$$

$$\begin{aligned} \Delta \lambda_5 = & \Delta \rho_5 - c \Delta \rho_5 + a \Delta \rho_2 (\Delta \rho_3 - c \Delta \rho_3) + a \Delta \rho_3 (\Delta \rho_2 - c \Delta \rho_2) - 2b \Delta \rho_2 (\Delta \rho_2 - c \Delta \rho_2) \\ & (-b \Delta \rho_4 + a \Delta \rho_3 - a \overline{\Delta \rho_2}^2 b) - 3[\Delta \rho_3 - c \Delta \rho_3 + 2(\Delta \rho_2 - c \Delta \rho_2) b \Delta \rho_2] \end{aligned}$$

$$(-b \Delta \rho_3 + a \Delta \rho_2) + K \Delta \lambda_4 b \Delta \rho_2,$$

and:

$$\Delta \mu_2 = \Delta m_2 - d \Delta \rho_2,$$

$$\Delta \mu_3 = \Delta m_3 - d \Delta \rho_3 + 2(\Delta m_2 - d \Delta \rho_2) b \Delta \rho_2,$$

$$\begin{aligned} \Delta \mu_4 = & \Delta m_4 - d \Delta \rho_4 + a \Delta \rho_2 (\Delta m_2 - d \Delta \rho_2) + 2(\Delta m_2 - d \Delta \rho_2) (b \Delta \rho_3 - a \Delta \rho_2) \\ & + (\Delta m_2 - d \Delta \rho_2) b^2 \overline{\Delta \rho_2}^2 + 3b \Delta \rho_2 [\Delta m_3 - d \Delta \rho_3 + 2(\Delta m_2 - d \Delta \rho_2) b \Delta \rho_2], \end{aligned}$$

$$\begin{aligned} \Delta \mu_5 = & \Delta m_5 - d \Delta \rho_5 + a \Delta \rho_2 (\Delta m_3 - d \Delta \rho_3) + a \Delta \rho_3 (\Delta m_2 - d \Delta \rho_2) \\ & - 2(\Delta m_2 - d \Delta \rho_2) (-b \Delta \rho_4 + a \Delta \rho_3 - a \overline{\Delta \rho_2}^2 b) - 3[\Delta m_3 - d \Delta \rho_3 + 2(\Delta m_2 - d \Delta \rho_2) b \Delta \rho_2] \\ & (-b \Delta \rho_3 + a \Delta \rho_2) + K \Delta \mu_4 b \Delta \rho_2, \end{aligned}$$

We see from these equations that the theorems already proved  
for the general case hold for the case where .

Case II. Projections from a line on a plane.

Our curve  $\mathcal{C}$  is given, in the coordinate system set up in the first part of this paper, by the equations:

$$(1) \quad y = f_1(x)$$

$$(2) \quad z = f_2(x)$$

$$(3) \quad w = f_3(x)$$

Equations (1) and (2) give us the projection of  $\mathcal{C}$  on the osculating three space from the fifth vertex of the corresponding homogeneous coordinate system. We designate this projection on the osculating three space as  $\mathcal{C}_1$ . Equation (1) gives us the projection of  $\mathcal{C}_1$  on the osculating plane from the fourth vertex of the corresponding homogeneous coordinate system. We designate this projection as  $\mathcal{C}_2$ . We now show that we may consider  $\mathcal{C}_2$  as the projection of  $\mathcal{C}$  from the line joining the fourth and fifth vertices of the corresponding homogeneous coordinate system on the osculating plane. Let the join of the fourth and fifth vertices be  $\ell$ .

We first show that such a projection exists. Consider a general point  $A$  on our  $\mathcal{C}$  curve.  $A$  and  $\ell$  determine a plane  $\alpha$ ; this plane meets the osculating plane in a point  $B$ , as it is a well known property of four dimensional space that two planes must meet in a point.\* Then as  $A$  moves along the curve  $\mathcal{C}$ ,  $B$  must describe a curve in the osculating plane; by definition this curve

---

\*If  $\alpha$  and the osculating plane met in a line, then all five vertices of our homogeneous coordinate system would be contained in a plane and a line having a point in common, that is, in a three space. This is, of course, impossible.

is the projection of  $\mathcal{C}$  from  $\ell$  on the osculating plane.

We now show that equation (1) represents the curve described by  $\mathcal{B}$ . Consider our point  $A$  on  $\mathcal{C}$ .  $A$  and the fifth vertex of the corresponding homogeneous coordinate system determine a line meeting the osculating three space in a point  $A'$ .  $A'$  and the fourth vertex of the corresponding homogeneous coordinate system determine a line  $m$  which meets the osculating plane in a point  $A''$ . We are to show that  $A''$  is  $\mathcal{B}$ .  $\ell$  and  $A$  are in  $\alpha$ . Then  $A'$  is in  $\alpha$ , since it is on the join of two points in  $\alpha$ . Similarly  $A''$  must be in  $\alpha$ , since it is on the join of  $A'$  and the fourth vertex, which is in  $\alpha$ . Then  $A''$  is in  $\alpha$  and in the osculating plane, likewise  $\mathcal{B}$  is in  $\alpha$  and in the osculating plane. Since  $\alpha$  and the osculating plane meet in but one point,  $A'' \equiv \mathcal{B}$ . Since  $A''$  describes the curve represented by equation (1), the demonstration is completed.

In order to study the projections from a line on the osculating plane of the two curves  $\mathcal{C}$  and  $\mathcal{C}'$  we use the equations of the curves obtained by a transformation of the fifth vertex of the homogeneous coordinate system to the point  $(a \ell c d 1)$ . Here also we are only interested in the  $x, y$  and  $x, z$  relations. These equations for  $\mathcal{C}$  are equations (11) and (12); for  $\mathcal{C}'$  they are (13) and (14). We reproduce them here, including also the coefficients of  $x^{u+4}$ , which heretofore we have not determined.

For  $\mathcal{C}$  :

$$11) Y = \lambda_2 X^2 + \lambda_3 X^3 + \dots + \lambda_n X^n + \lambda_{n+1} X^{n+1} + \lambda_{n+2} X^{n+2} + \dots,$$

$$12) Z = \mu_3 X^3 + \mu_4 X^4 + \dots + \mu_n X^n + \mu_{n+1} X^{n+1} + \mu_{n+2} X^{n+2} + \dots,$$

where:

$$11-a) \lambda_{n+1} = \rho_{n+1} - c\rho_{n+1} + M_{n+1}^{IV},$$

$$11-b) \lambda_{n+2} = \rho_{n+2} - c\rho_{n+2} + 2b_2 \rho_{n+1} + M_{n+2}^{IV},$$

$$11-c) \lambda_{n+3} = \rho_{n+3} - c\rho_{n+3} - a b_2 \rho_{n+1} + 2b_2 \rho_{n+2} + 3b_3 \rho_{n+1} + M_{n+3}^{IV},$$

$$11-d) \lambda_{n+4} = \rho_{n+4} - c\rho_{n+4} + b_3 a \rho_{n+1} + b_2 a \rho_{n+2} + 2b_2 \rho_{n+3} - 2b_2 a \rho_{n+2} + 3b_3 \rho_{n+2} - 3a b_3 \rho_{n+1} + K(b_4 - c b_4) \rho_{n+1} + M_{n+4}^{IV},$$

and:

$$12-a) \mu_{n+1} = m_{n+1} - d\rho_{n+1} + P_{n+1}^{II},$$

$$12-b) \mu_{n+2} = m_{n+2} - d\rho_{n+2} + P_{n+2}^{II},$$

$$12-c) \mu_{n+3} = m_{n+3} - d\rho_{n+3} + 3m_3 \rho_{n+1} + P_{n+3}^{II},$$

$$12-d) \mu_{n+4} = m_{n+4} - d\rho_{n+4} + 3m_3 \rho_{n+2} - 2m_3 c\rho_{n+1} - K(m_4 - d\rho_{n+1})(-b\rho_{n+1}) + P_{n+4}^{II},$$

where  $K$  is a numerical constant.

For  $\mathcal{C}'$  :

$$13) Y = \lambda_2 X^2 + \lambda_3 X^3 + \dots + \lambda_n X^n + \lambda'_{n+1} X^{n+1} + \lambda'_{n+2} X^{n+2} + \dots,$$

$$14) Z = \mu_3 X^3 + \mu_4 X^4 + \dots + \mu_n X^n + \mu'_{n+1} X^{n+1} + \mu'_{n+2} X^{n+2} + \dots,$$

where:

$$13-a) \lambda'_{n+1} = \rho'_{n+1} - c\rho'_{n+1} + M_{n+1}^{IV},$$

$$13-b) \lambda'_{n+2} = \rho'_{n+2} - c\rho'_{n+2} + 2b_2 \rho'_{n+1} + M_{n+2}^{IV},$$

$$13-c) \lambda'_{n+3} = \rho'_{n+3} - c\rho'_{n+1} - a\ell_2\rho'_{n+1} + 2\ell_2\ell\rho'_{n+2} + 3\ell_3\ell\rho'_{n+1} + M_{n+3}^{\text{II}}$$

$$13-d) \lambda'_{n+4} = \rho'_{n+4} - c\rho'_{n+4} + \ell_2 a\rho'_{n+1} + \ell_2 a\rho'_{n+2} + 2\ell_2\ell\rho'_{n+3} - 2\ell_2 a\rho'_{n+2} + 3\ell_3\ell\rho'_{n+2} - 3\ell_3 a\rho'_{n+1} + K(\ell_4 - c\rho_4)\ell\rho'_{n+1} + M_{n+4}^{\text{IV}}$$

and:

$$14-a) \mu'_{n+1} = \mu'_{n+1} - d\rho'_{n+1} + P_{n+1}''$$

$$14-b) \mu'_{n+2} = \mu'_{n+2} - d\rho'_{n+2} + P_{n+2}''$$

$$14-c) \mu'_{n+3} = \mu'_{n+3} - d\rho'_{n+3} + 3\ell_3\ell\rho'_{n+1} + P_{n+3}''$$

$$14-d) \mu'_{n+4} = \mu'_{n+4} - d\rho'_{n+4} + 3\ell_3\ell\rho'_{n+2} - 2\ell_3 a\rho'_{n+1} - K(\ell_4 - d\rho_4)(-\ell\rho'_{n+1}) + P_{n+4}''$$

We first show that by a proper choice of the coordinates of our fourth and fifth vertices of the corresponding homogeneous coordinate system the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from the line  $\ell$  on the osculating plane may be made of order of contact  $n+4$ .

As before, we choose our fifth vertex such that  $\lambda_{n+1} = \lambda'_{n+1}$  and

$\mu_{n+1} = \mu'_{n+1}$ . Then by our first case the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  on the osculating three-space from the fifth vertex <sup>are</sup> of order of contact  $n+1$ . We designate these projections on the osculating three-space as  $\mathcal{C}_1$  and  $\mathcal{C}'_1$  respectively. Now, from the

theorems stated at the beginning of this paper we may choose our fourth vertex in such a manner that the projection <sup>from</sup> it of  $\mathcal{C}$  and

$\mathcal{C}'$  on the common osculating plane are of order of contact three greater than  $\mathcal{C}_1$  and  $\mathcal{C}'_1$ . We denote these projections by  $\mathcal{C}_2$  and  $\mathcal{C}'_2$  respectively; they have order of contact  $n+4$ . Since  $\mathcal{C}_2$  and  $\mathcal{C}'_2$  may be considered as the projections from the line

$\ell$  on the common osculating plane, we may say that there exists at least one line  $\ell$  which has the property that the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from the line on the osculating plane have contact of order  $n+4$ .

We shall study the line  $\ell$ . It is the join of the fifth and fourth vertices of the corresponding new homogeneous coordinate system. The only restriction we placed upon the fifth vertex was that it should be in the principal tangent plane, therefore one point of  $\ell$  is any point of the principal tangent plane. The restriction we place upon the fourth vertex is that it is the principal point of the two curves  $\mathcal{C}$  and  $\mathcal{C}'$ . Thus our line of projection  $\ell$  meets the principal tangent plane of the two curves  $\mathcal{C}$  and  $\mathcal{C}'$  in a point  $R$ , and contains the principal point,  $P$ , of the two curves which are the projections of  $\mathcal{C}$  and  $\mathcal{C}'$  from  $R$  on the common osculating three-space of the two curves. We thus have a correspondence between the points of our principal tangent plane and certain points of our osculating three-space.

We shall study more closely these points in our common osculating three space. In order to do this we reproduce the analytic results of Stouffer's work on the three dimensional case, treated in a paper already cited. Given two three dimensional space curves  $\mathcal{C}$  and  $\mathcal{C}'$  of order of contact  $n$  at a point, their



equations may be put in the form:

For  $C_1$  :

$$\alpha) y = \lambda_2 x^2 + \lambda_3 x^3 + \dots + \lambda_n x^n + \lambda_{n+1} x^{n+1} + \dots,$$

$$\beta) z = s_2 x^2 + s_3 x^3 + \dots + s_n x^n + s_{n+1} x^{n+1} + \dots,$$

For  $C'_1$  :

$$\alpha') y = \lambda_2 x^2 + \lambda_3 x^3 + \dots + \lambda_n x^n + \lambda'_{n+1} x^{n+1} + \dots,$$

$$\beta') z = s_2 x^2 + s_3 x^3 + \dots + s_n x^n + s'_{n+1} x^{n+1} + \dots,$$

We may obtain the principal line, plane, and point of the two curves by the variation of the coordinates, in the corresponding homogeneous coordinate system, of the point  $(j, k, l, 1)$ .

We denote by  $\Delta \lambda_i$  the difference  $\lambda_i - \lambda'_i$ ; similarly  $\Delta s_i = s_i - s'_i$ .

Then the principal tangent plane of  $C_1$  and  $C'_1$  is obtained if we assign to  $l$  the value:

$$l = \frac{\Delta \lambda_{n+1}}{\Delta s_{n+1}}.$$

The principal tangent line is obtained by assigning to  $k$  the value:

$$k = \frac{l \Delta s_{n+2} - \Delta \lambda_{n+2}}{2 \lambda_2 \Delta s_{n+1}}.$$

The principal point is obtained by assigning to  $j$  the value:

$$j = \frac{l \Delta s_{n+3} - \Delta \lambda_{n+3} - 2 \lambda_2 k \Delta s_{n+2} - 3 \lambda_3 k \Delta s_{n+1} + 3 s_2 l \Delta s_{n+1}}{-\lambda_2 \Delta s_{n+1}}.$$

We wish to study the variations of  $l$ ,  $k$ , and  $j$  with  $a$ . To do this we express the coefficients of  $C_1$  and  $C'_1$  as functions of  $a$ , regarding all other quantities as constants: We adopt the

notation  $\Delta \lambda_i = \lambda_i - \lambda_i'$ ;  $\Delta \mu_i = \mu_i - \mu_i'$ . We also establish the correspondence between the coefficients of equations (//) and (12), (13) and (14) and equations ( $\alpha$ ) and ( $\beta$ ), ( $\alpha'$ ) and ( $\beta'$ ) respectively. We denote this correspondence by  $\equiv$

$$\Delta \lambda_{n+1} \equiv \Delta \lambda_{n+2} = K_1,$$

$$\Delta S_{n+1} \equiv \Delta \mu_{n+2} = K_2,$$

$$\therefore l = K_3,$$

$$\Delta \lambda_{n+2} \equiv \Delta \lambda_{n+3} = K_4 + a K_5,$$

$$\Delta S_{n+2} \equiv \Delta \mu_{n+3} = K_6,$$

$$\therefore h = K_7 a + K_8,$$

$$\Delta \lambda_{n+3} \equiv \Delta \lambda_{n+4} = K_9 a + K_{10},$$

$$\Delta S_{n+3} \equiv \Delta \mu_{n+4} = K_{11} a + K_{12},$$

$$\therefore j = K_{13} a + K_{14},$$

where  $K_i^0$  is a constant with respect to  $a$ .

We summarize:

$$l = S_1,$$

$$h = S_2 a + S_2',$$

$$j = S_3 a + S_3',$$

Since  $l$  is a constant with respect to  $a$ ; since  $a$  is the  $x_1$  coordinate of our point of projection, and since we have already shown that the variations of  $a$  give us a line on the principal tangent plane of  $C$  and  $C'$  through the point of contact of  $C$  and  $C'$ ; we may conclude that the projections of  $C$  and  $C'$  on the common osculating three-space have the same principal tangent plane, if the point of projection is on a line, in the principal tangent plane of  $C$  and  $C'$  and containing the point of contact of the two curves. Thus as  $R$  moves in a straight line through the point of contact of the two curves,  $P$  moves in a plane curve. We designate this curve as  $P$ , we shall derive its form.

We form the equations of the point  $P$  whose coordinates are  $(jkl)$  where  $l$  is constant and  $j$  and  $k$  are linear functions of  $a$ . Then we may express the point by the following parametric equations:

$$a) \rho x_5 = 0,$$

$$b) \rho x_4 = 1,$$

$$c) \rho x_3 = l',$$

$$d) \rho x_2 = S_2 a + S_2',$$

$$e) \rho x_1 = S_3 a + S_3',$$

where  $a$  is the parameter. Expressing these equations in homogeneous form by means of (6), and eliminating the parameter between (d) and (e), we obtain:

$$f) \quad \rho x_5 = 0.$$

$$g) \quad M_1 x_2 + M_2 x_4 = 0.$$

$$h) \quad L_1 x_2 + L_2 x_3 + L_3 x_4 = 0,$$

where  $M_i$  and  $L_i$  are constants. Equations (f), (g) and (h) are the equations of our curve  $P$ . We conclude that our curve  $P$  is in general a straight line not containing the point of contact of  $C$  and  $C'$ .

We study now the projections from the line  $\ell$  on the common osculating plane of the two curves when the fourth vertex is any given point in the common osculating three-space. Equations (11) and (13) show us that by assigning a certain value to  $c$  we may bring the order of contact of the two curves up to  $h+1$ . Thus for every point  $Q$  of our common osculating three-space there exists a unique three-space  $\omega$  that has the property that the projections of  $C$  and  $C'$  from the join of any point in the three-space on the common osculating plane of  $C$  and  $C'$  have contact of order  $h+1$ . That the principal tangent plane of  $C$  and  $C'$  is contained in  $\omega$  is obvious both by analytical and geometrical reasoning.  $\omega$  also contains the vertices (10000), (01000), and (00010) of the old homogeneous coordinate system.

If we next assign a certain value to  $\beta$  such that

$\lambda_{n+2} = \lambda'_{n+2}$ , the order of contact of  $\mathcal{C}_2$  and  $\mathcal{C}'_2$  is  $n+2$ .

These values of  $c$  and  $\beta$  determine a plane,  $\beta$ , contained in  $\omega$ . Since  $\beta$  is described by the variations of  $a$  and  $d$  it contains the points (10000) and (00010) of the corresponding homogeneous coordinate system. If we consider  $Q$  as the center of a bundle of lines, possessing the property that the projections from them of  $\mathcal{C}$  and  $\mathcal{C}'$  on the osculating plane are of order of contact  $n+1$ , then we may consider  $\beta$  as a certain planar pencil of lines, being a part of the lines of the bundle, and possessing the property that the projections from them are of order of contact  $n+2$ .

If we next assign a certain value to  $a$  such that

$\lambda_{n+3} = \lambda'_{n+3}$ , the order of contact of  $\mathcal{C}_2$  and  $\mathcal{C}'_2$  is  $n+3$ .

The line determined by the given values of  $a$ ,  $\beta$ , and  $c$  contains the fourth vertex ( $Q$ ) of the homogeneous coordinate system, thus it is a line of the bundle of lines. The line possesses the property that the projection from it of  $\mathcal{C}$  and  $\mathcal{C}'$  on the common osculating plane is of order of contact  $n+3$ . This work is chiefly of interest due to its ready analogy with the three dimensional case.

We may show that our theorems of Case II hold for the special cases outlined in Case I. We outline the proof here.

(1)  $\rho_{n+1} = \rho'_{n+1}$  . Equations (22), (23), (24), (25) tell us that our line  $\ell$  in this case is the join of the fourth and fifth vertices, however it is the fourth vertex that is confined to the principal plane, and the fifth vertex that is determined by points of the fourth vertex. We may also develop the identical theory as before in considering the projections from  $\ell$  where the fifth vertex is any given point.

(2)  $n=1, n=2$  . In these cases the coefficients of the transformed equations are similar to those of the general equations, therefore all the theorems of Case II follow immediately.

We have shown that there exists a family of lines from which the projections of  $\ell$  and  $\ell'$  on the common osculating plane are of order of contact  $n+4$  . We ask the question: "May it be possible to choose a line such that the corresponding projections are of order of contact greater than  $n+4$  ?" We shall discuss this possibility.

If we make a transformation of coordinates that involves only the changing of our fifth vertex to a point  $(abcd1)$  and the changing of our fourth vertex to a point  $(efg10)$ , our resulting

$X, Y$  relations for  $\ell$  and  $\ell'$  are the projections of  $\ell$  and  $\ell'$  from the join of  $(abcd1)$  and  $(efg10)$  on the osculating plane.

It may then be possible to choose our coordinates such that the order of contact of the projections is greater than  $n+4$ .

Our transformation has the form:

$$X = \frac{x - fz + (fd - b)w}{1 - \epsilon z + (\epsilon d - a)w},$$

$$Y = \frac{y - gz + (gd - c)w}{1 - \epsilon z + (\epsilon d - a)w},$$

Let  $h = (\epsilon d - a)$ ,  $j = (fd - b)$ ,  $k = (gd - c)$ .

Then our transformation has the form:

$$X = (x - fz + jw) [1 + (\epsilon z - hw) + (\epsilon z - hw)^2 + \dots],$$

$$Y = (y - gz + kw) [1 + (\epsilon z - hw) + (\epsilon z - hw)^2 + \dots],$$

We notice that in our transformation we have only 6 distinct functions of the coordinates of our new fourth and fifth vertices, hence the order of contact of  $\mathcal{C}_2$  and  $\mathcal{C}'_2$  cannot in general be increased more than six. At present, we can say little more than this. The actual transformation of our  $\mathcal{C}$  and  $\mathcal{C}'$  curves to the new coordinate system is of such complexity that it cannot be completed in this paper. We may only say that the conditions that must be satisfied in order that we may assign values to the coordinates of our fourth and fifth vertices to bring the order of contact of the two projections up to  $n+6$  seem to be the existence of solutions to a set of six simultaneous equations in six variables, the degree of some of the equations being three and possibly four

and the number of terms in at least one of the equations being at least 60.